How many cards do you need before you have to have a set?

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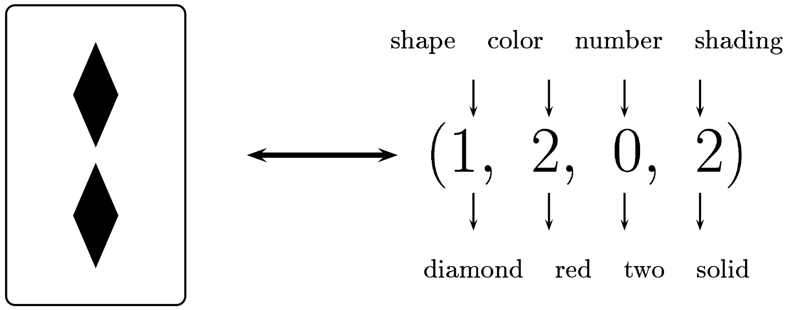
A mathematician would innocently interpret the title question in terms of set theory and might answer something like "zero or more." But readers of this issue will immediately interpret the question as referring to the card game Set, and understand the question to refer to the size of the smallest collection of Set cards for which the probability of having a "set" is equal to one.

Before describing what is known about this (and related) questions, it is convenient to convert this into a more mathematical question. After playing the game a bit you will probably be convinced that the game has considerable mathematical structure, and I want to first convince you that the game is actually about lines in a finite 4-dimensional space!

Set for the blind

The card came Set relies heavily on sight. Suppose that you want to design a collection of cards that can be played by blind people that relies on the sense of touch. Elaborate ways to do this are easy to imagine, but we will imagine that everything will be converted into numbers that will be represented by zero, one, or two raised bumps.

Each Set card has four categories (color, shape, number, and shading) each of which has three possible values (e.g., red, green, or purple for the color category). We arbitrarily order the categories and arbitrarily number the values in each category 0, 1, or 2. The result is that we have a one-to-one correspondence between Set cards and 4-tuples whose components are in $\{0, 1, 2\}$. For instance, a card with two red, solid diamonds might correspond to $(1, 2, 2, 0)$ if the categories are, in order, shape, color, number, and shading, and the values have the indicated correspondences.



(As a practical matter, the cards would have to come equipped with orientation information, so that the user could feel the order of categories and then feel the number of bumps in each of the respective location; one could also imagine intersecting conflicts when players were feeling the same card...)

Under this correspondence between cards and 4-tuples, the game consists of a collection of cards, each of which is an ordered 4-tuple whose components are chosen from $\{0, 1, 2\}$. We will call a typical card a "vector" and write it in the form

$$v = (v_1, v_2, v_3, v_4), \quad v_1, v_2, v_3, v_4 \in \{0, 1, 2\}.$$

Since each of the 4 coordinates has 3 possible values, there are 81 cards all together.

Three such vectors (cards) are a "set" if for each component the values are either all alike or all different. For instance, the three triples $(0, 1, 2, 2)$, $(1, 1, 1, 2)$, and $(2, 1, 0, 2)$ are a set.

Sets are lines

The use of the word "set" is overloaded in the sense that it has three meanings (the mathematical meaning, name of the game, a winning triple in the game). From now on, we will refer to the game in upper case, and refer to a winning triple of cards as a "line" (for reasons that will gradually emerge).

The key to a geometric interpretation of the game lies in using a finite arithmetic, namely, arithmetic on $\{0, 1, 2\}$ modulo 3, which we briefly review.

Arithmetic modulo 3. Arithmetic is done as usual except that the result of any computation is replaced with its remainder after dividing by 3. This enables us to remain entirely within the set $\{0, 1, 2\}$. In this world, $2 + 2 = 1$ since the remainder of 4 when divided by 3 is 1. Similarly we have

$$2+2+2=0, \quad 2+2+2+2=2, \quad 2\cdot 2=1, \quad 2\cdot 2\cdot 2=2, \quad 2\cdot 2\cdot 2\cdot 2\cdot 2=1.$$

The key observation is that the condition that three elements of $\{0,1,2\}$ are "all alike" or "all different" is equivalent to saying that the add to 0.

Fact 1: If x, y, and z are elements of $\{0,1,2\}$, then $x+y+z=0$, using clock arithmetic modulo 3, if and only if $x=y=z$ or $\{x,y,z\}=\{0,1,2\}$.

(The reader should verify this Fact!)

In Set, we have to apply this idea to all coordinates at once, so we remind the reader that vectors can be added by adding their components. For instance, sum of $(2,1,0,2)$ and $(1,1,1,2)$ is

$$(2,1,0,2)+(1,1,1,2)=(0,2,1,1)$$

(where of course all arithmetic is done modulo 3). More formally, we could define

$$u+v=(u_1,u_2,u_3,u_4)+(v_1,v_2,v_3,v_4)=(u_1+v_1,u_2+v_2,u_3+v_3,u_4+v_4).$$

Combining everything that's been done, we can now say that three vectors (cards) u, v, and w are a line if and only if $u+v+w=(0,0,0,0)$.

Actually, in this context the more official definition of a line (by which we mean "affine line" in the usual geometric terminology) is that a line consists of three vectors of the form u, $u+v$, and $u-v$ where u and v are vectors (with v nonzero) and subtraction is defined in the obvious way. The reader should attempt to verify that this is equivalent to the "all like or all different" definition or the "sum to zero" result just obtained.

Just as in ordinary geometry, any two distinct cards determine a unique line. Good Set players already know this: give two cards, their is a unique third card that forms a line with them. In fact, the reader should verify that if u and v are distinct vectors then the third vector that forms a line with them is

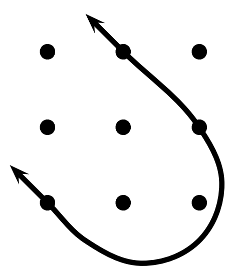
$$w=\overline{u+v}$$

where the overbar indicates that the numbers 1 and 2 are interchanged in the components (i.e., a 1 becomes a 2 and vice versa).

Set in all dimensions

Another upshot of this geometric view of the game is that, from a mathematical point of view, there is nothing very special about the number of components: the game makes sense in n-dimensional space, where there are 3^n cards and n categories.

For instance, for $n = 2$ there are two categories and we could visualize the entire 9-card deck as a coordinate plane in which each coordinate can take on only 3 values. There are 12 lines in this game: the three vertical lines, three horizontal lines, three lines of slope 1, and three lines of slope 2. In the drawing below, one of the lines of slope 2 is indicated.



Line-free sets

What is the maximum size of a line-free set? This is a special case of a question in finite geometry. Namely, in various finite geometries (of which the one that we are looking at here is a very special case) what is the largest set not containing three collinear points? Such sets are called "arcs." This idea was originally motivated by a question in the theory of error-correcting codes, which are used, for instance, to encode data in redundant fashions on CD's so that minor damage does not cause any data to be lost.

In any event, let's start by looking at the two-dimensional case. It is easy to find a set of 4 vectors that is line-free; perhaps the simplest is the set $\{(0,0),(1,0),(0,1),(1,1)\}$ of all 0-1 vectors whose components are 0 or 1. A little struggling shows that this is the largest line-free set.

In fact, in the n-dimensional case there is always a line-free set with 2^n points: the set of all 0-1 vectors. (The reader should convince him or herself that this set is in fact line-free. Do it now!)

In 3-dimensions a little experimentation one can do a little better: there is a 9-element line-free set. (The reader should now find one.) A little further experimentation should convince the reader that any 10-element set in 3 dimensions contains a line. Can you prove this?

20 cards without a Set

In the case of four dimensions, we are dealing with the game Set, and the reader who has played the game will have some intuition. Although 12-element sets usually contain lines, the reader will no doubt have run across 12 and even 15-element collections that do not contain lines. It may not be entirely clear as to how large a set one needs to be **certain** of having a set. It turns out that the number is 21.

Put another way, the largest possible line-free set contains 20 cards. One such set is fairly easy to describe. Namely, following the notation of Calderbank and Fishburn in [1], we let '$*$' denote either 1 or 2, and let, for instance $(*,*,0,0)$ denote all four vectors whose first components are 1 or 2 and last two components are zero. Then the reader can verify, with some worth that the set

$$(*,*,0,0)$$
$$(*,0,*,0)$$
$$(*,0,0,*)$$
$$(0,*,*,*)$$

is a line-free set containing 20 vectors. (The reader should, with some ingenuity, be able to easily prove that this set is in fact line-free.) Apparently the first proof of the fact that this is the largest possible size was given in [3] was given in [3] (actually, the proof there considered the slightly more general situation of "projective" geometry). There are other 20-element line-free sets, that are "genuinely different" from the above set, in the sense that they can not be obtained from this set by permuting coordinates or permuting the symbols $\{0,1,2\}$. As a genuinely challenging exercise, the reader might try to find such a set.

The fifth dimension

What about the 5-dimensional case? The answer isn't known! The situation is actually interesting. Due to a combination of favorable circumstances, the paper [2] was able to find the largest possible set in 5-dimensional "projective space" (whatever that is) by using some elegant facts from the theory of error-correcting codes. However, this does not seem to resolve the situation in the case of the "affine" geometry. For the moment, this question seems slightly beyond the reach of computers, and slightly beyond the reach of the human mind. My own intuition is that some combination will be able to resolve the question in the near future.

$$(0,1,0,2,0)(0,2,0,2,2)(1,0,0,2,0)(2,0,0,2,1)(1,1,0,1,1)$$
$$(2,2,0,1,1)(1,2,0,1,1)(2,1,0,1,0)(0,0,1,0,1)(0,0,1,1,2)$$
$$(0,1,1,0,2)(0,1,1,1,0)(0,2,1,0,1)(0,2,1,1,0)(1,0,1,0,0)$$
$$(1,0,1,1,2)(2,0,1,2,2)(2,0,1,1,1)(1,1,1,0,0)(1,1,1,2,2)$$
$$(2,2,1,0,0)(2,2,1,1,2)(1,2,1,2,2)(1,2,1,1,0)(2,1,1,0,0)$$
$$(2,1,1,2,1)(0,0,2,2,0)(0,0,2,1,1)(0,1,2,2,1)(0,1,2,1,2)$$
$$(0,2,2,0,2)(0,2,2,2,1)(1,0,2,0,2)(1,0,2,2,1)(2,0,2,0,2)$$
$$(2,0,2,1,0)(1,1,2,0,1)(1,1,2,1,2)(2,2,2,0,2)(2,2,2,2,0)$$
$$(1,2,2,0,1)(1,2,2,2,0)(2,1,2,0,1)(2,1,2,2,2)(0,0,0,0,0)$$

Could you tell what this is? A line-free collection of 45 cards in 5-dimensional space?

References

1. A.R. Calderbank, P.C. Fishburn, *Maximal three-independent subsets of* $\{0,1,2\}^n$. Des. Codes Cryptogr. 4 (1994), no. 3, 203–211.
2. Hill, Raymond Caps and codes. Discrete Math. 22 (1978), no. 2, 111–137.
3. Giuseppe Pellegrino, *Sul massimo ordine delle calotte in* $S_{4,3}$. Matematiche (Catania) 25 (1971), 149–157.